SECOND-ORDER SHAPE DERIVATIVES ALONG NORMAL TRAJECTORIES, GOVERNED BY HAMILTON-JACOBI EQUATIONS.

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Computing derivatives with respect to a shape - a variable subset of $\mathbb{R}^d$ - is essential for shape optimization. Following the work of Murat and Simon [5] one can define a variation of a domain as a perturbation by a regular mapping : any domain Ω "close" to Ω_0 can be seen as $\Omega = \big(\mathrm{Id} + \theta\big)(\Omega_0)$, where θ is small regular displacement field, and Id is the identity operator on $\mathbb{R}^d$. This allows to get Fréchet derivatives of function depending on Ω [3, 6]. On the other hand, one can consider a moving domain $[0, \tau[\ni t \mapsto \Omega_t$, and then obtain Fréchet derivatives of function depending on Ω. Zolesio and co-workers [2] choose $\Omega_t = X_V\big(t, \Omega_0\big)$, where $X_V : [0, \tau[\times\mathbb{R}^d$ is the maximal flow of a regular vector field $V \in C^k\big(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d\big) : \partial_t X_V(t, x) = V\big(t, X_V(t, x)\big)$ with $X_V(0, x) = x$. The derivation of $[0, \tau[\ni t \mapsto \Omega_t$ with respect to t gives another concept of shape derivative. In both contexts, the structure of the second order shape derivatives is well described [2, 3].

The goal of this work is to derive along time trajectories - similarly to this approach - in the case of the level-set method, when the vector field is aligned with the outer normal of the shape. Let φ_0 be a level-set function representing Ω_0 :

$$\begin{cases} x \in \Omega_0 & \text{if } \varphi_0(x) < 0, \\ x \in \partial\Omega_0 & \text{if } \varphi_0(x) = 0, \\ x \in \mathbb{R}^d\backslash\overline{\Omega_0} & \text{if } \varphi_0(x) > 0. \end{cases} \tag{0.1}$$

The boundary of Ω_0 is given by the level set $\{x \in \mathbb{R}^d \mid \varphi_0(x) = 0\}$. Given a regular vector field $v \in C^k\big(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}\big)$, let $\varphi_v \in C^k\big([0, \tau[\times\mathbb{R}^d\big)$ be a classical solution of the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t\varphi_v(t, x) + v(t, x)\,|\nabla_x\varphi_v(t, x)| = 0, \\ \varphi_v(0, x) = \varphi_0(x). \end{cases} \tag{0.2}$$

For $0 \le t < \tau$ we define Ω_t as being the set of negative values of $\varphi_v(t, \cdot)$: $\Omega_t = \{x \in \mathbb{R}^d \mid \varphi_v(t, x) < 0\}$. The present work focuses on the shape derivatives in this particular framework, with the help of the bicharacteristic method for solving the Hamilton-Jacobi equation [4]. At the end, this leads to a second order derivative which has a different structure from those of the derivatives obtained in the previous contexts.

References

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